

CONJUGATING HOMEOMORPHISMS TO UNIFORM HOMEOMORPHISMS

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ABSTRACT. Let $H(X)$ denote the group of homeomorphisms of a metric space X onto itself. We say that $h \in H(X)$ is conjugate to $g \in H(X)$ if $g = fhf^{-1}$ for some $f \in H(X)$. In this paper, we study the questions: When is $h \in H(X)$ conjugate to $g \in H(X)$ which is a uniform homeomorphism or can be extended to a homeomorphism \tilde{g} on the metric completion of X ? Typically for a complete metric space X , we prove that $h \in H(X)$ is conjugate to a uniform homeomorphism if H is uniformly approximated by uniform homeomorphisms. In case $X = \mathbf{R}$, we obtain a stronger result showing that every homeomorphism on \mathbf{R} is, in fact, conjugate to a smooth Lipschitz homeomorphism. For a noncomplete metric space X , we provide answers to the existence of \tilde{g} under several different settings. Our results are concerned mainly with infinite-dimensional manifolds.

0. INTRODUCTION

All spaces considered in this paper are metric spaces. Given a metric space $X = (X, d)$, let $H(X)$ denote the group of homeomorphisms of X onto itself. We say $h \in H(X)$ is *conjugate* to $g \in H(X)$ if there is an $f \in H(X)$ such that $g = fhf^{-1}$. Let $H_U(X)$ denote the set consisting of all $h \in H(X)$ such that both h and h^{-1} are uniformly continuous. Members of $H_U(X)$ will be referred to as *uniform homeomorphisms*. Let \tilde{X} be the metric completion of X . In this paper, we study the following questions:

(A) When is an $h \in H(X)$ conjugate to a $g \in H_U(X)$?

(B) When is an $h \in H(X)$ conjugate to a $g \in H(X)$ which can be extended to a $\tilde{g} \in H(\tilde{X})$?

Question (B) was raised in [AB] (and [AK, CSQ 4]) in the setting of $X = s = (-1, 1)^\infty$ and $\tilde{X} = Q = [-1, 1]^\infty$. Since any $g \in H_U(X)$ can be extended to a $\tilde{g} \in H(\tilde{X})$ (in fact, to a $\tilde{g} \in H_U(\tilde{X})$), (A) implies (B). In general, (B) does not imply (A). However, if \tilde{X} is compact, then (B) is equivalent to (A).

In §1, we present the special case $X = \mathbf{R}$, the real line with Euclidean metric. Our result shows that, in fact, every homeomorphism on \mathbf{R} is conjugate to a

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smooth Lipschitz homeomorphism with $\text{bilip} \leq 1 + \varepsilon$ for any given $\varepsilon > 0$ (see §1 for definition).

In §2, we provide an answer for Question (A) in case X is a complete metric space. Our main result of the section (Theorem 2.1) shows that $h \in H(X)$ is conjugate to a $g \in H_U(X)$ provided $h \in \overline{H_U(X)}$, that is, h is uniformly approximated by uniform homeomorphisms. As a corollary, we obtain a similar result in case X is the interior of a complete metric n -manifold \tilde{X} . In general, we cannot weaken the condition $h \in \overline{H_U(X)}$. More specifically, in Example 2.4 we construct a uniformly continuous homeomorphism h on a complete metric, connected 2-manifold X with $\partial X \neq \emptyset$ such that h is not conjugate to any uniform homeomorphism on X .

In §3, we mainly consider finite- or infinite-dimensional manifold cases. Our results answer Question (B) in several different settings. For example, we show that if X is the interior of an n -manifold \tilde{X} , $n \neq 4, 5$, and if $h \in H(X)$ extends to a proper map of \tilde{X} then h is conjugate to a $g \in H(X)$ which extends to a $\tilde{g} \in H(\tilde{X})$ (Theorem 3.3). In general, we cannot eliminate the condition that h extends to a proper map of \tilde{X} . In Example 3.4 we construct a homeomorphism $h: \dot{B}^n \rightarrow \dot{B}^n$, $n \geq 2$, where \dot{B}^n is the interior of the n -ball B^n , such that h is not conjugate to any uniform homeomorphism (hence not extendible to B^n).

In §4, we consider the stable cases for infinite-dimensional manifolds where the given homeomorphisms are of the form (or conjugate to) $h \times \text{id}$. Roughly speaking, we show that stable homeomorphisms are always conjugate either to uniform homeomorphisms or to homeomorphisms extendible to its completion.

Finally in §5, we raise various questions.

1. LIPSCHITZ CONJUGATION OF HOMEOMORPHISMS OF \mathbf{R}

Let (X_1, d_1) and (X_2, d_2) be metric spaces. A bijective map $f: X_1 \rightarrow X_2$ is called a *Lipschitz homeomorphism* if there exists a real number $L \geq 1$ such that

$$L^{-1}d_1(x, y) \leq d_2(f(x), f(y)) \leq Ld_1(x, y)$$

for all $x, y \in X_1$. Let $\text{bilip}(f)$ denote the least such constant L for f . Our main result in this section is

1.1. Theorem. *Given $\varepsilon > 0$ and a homeomorphism h of \mathbf{R} onto itself, then h is conjugate to a smooth Lipschitz homeomorphism g with $\text{bilip}(g) \leq 1 + \varepsilon$.*

We start with the two special cases: fixed point free homeomorphisms and periodic homeomorphisms.

1.2. Proposition. *Any fixed point free homeomorphism of \mathbf{R} is conjugate to the translation $\tau: \mathbf{R} \rightarrow \mathbf{R}$ defined by $\tau(x) = x + 1$.*

Proof. Let $h \in H(\mathbf{R})$ be fixed point free. Then (1) $h(x) > x$ for all $x \in \mathbf{R}$ or (2) $h(x) < x$ for all $x \in \mathbf{R}$. In case (1), $h^n(0) \rightarrow \pm\infty$ as $n \rightarrow \pm\infty$, and in case (2), $h^n(0) \rightarrow \mp\infty$ as $n \rightarrow \pm\infty$. Let $f_0: [0, h(0)] \rightarrow [0, 1]$ (replace

$[0, h(0)]$ by $[h(0), 0]$ in case (2)) be a homeomorphism such that $f_0(0) = 0$ and $f_0(h(0)) = 1$. Extend f_0 to a homeomorphism $f: \mathbf{R} \rightarrow \mathbf{R}$ by

$$f([h^n(0), h^{n+1}(0)]) = \tau^n f_0 h^{-n} \quad \text{for each } n \in \mathbf{Z}$$

(replace $[h^n(0), h^{n+1}(0)]$ by $[h^{n+1}(0), h^n(0)]$ in case (2)). As easily observed, $fh = \tau f$, so h is conjugate to τ . \square

Hereafter, let $r: \mathbf{R} \rightarrow \mathbf{R}$ denote the reflection $r(x) = -x$.

1.3. Proposition. *Any periodic homeomorphism of \mathbf{R} is conjugate to the reflection r .*

Proof. Let $h \in H(\mathbf{R})$ be periodic. Then clearly $h^2 = \text{id}$ and h has exactly one fixed point $c \in \mathbf{R}$. Hence $x \leq c$ if and only if $h(x) \geq c$. We define a homeomorphism $f: \mathbf{R} \rightarrow \mathbf{R}$ as follows:

$$f(x) = \begin{cases} x - c & \text{for } x \geq c, \\ c - h(x) & \text{for } x \leq c. \end{cases}$$

We can easily check that $fh(x) = -f(x) = rf(x)$ for each $x \in \mathbf{R}$. Thus h is conjugate to r . \square

If $h \in H(\mathbf{R})$ is decreasing (i.e., order reversing), then h has exactly one fixed point $c \in \mathbf{R}$. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be the homeomorphism defined as in the above proof and let $g = fhf^{-1}$. Then $g(0) = 0$ and $g(x) = -x$ whenever $h^2 f^{-1}(x) = f^{-1}(x)$. Thus we have proved the following lemma, which will be needed in the proof of Theorem 1.1.

1.4. Lemma. *If $h \in H(\mathbf{R})$ is decreasing (i.e., order reversing), then h is conjugate to a $g \in H(\mathbf{R})$ such that $g(0) = 0$ and $g(x) = -x$ whenever x is a periodic point of g .*

We now state several lemmas which will also be needed in the proof of Theorem 1.1. The next lemma follows from the mean value theorem and the inverse function theorem.

1.5. Lemma. *A smooth map $g: \mathbf{R} \rightarrow \mathbf{R}$ is a Lipschitz homeomorphism if and only if $g'(x)$ and $g'(x)^{-1}$, $x \in \mathbf{R}$, are bounded. Furthermore*

$$\text{bilip}(g) = \max \left\{ \sup_{x \in \mathbf{R}} |g'(x)|, \sup_{x \in \mathbf{R}} |g'(x)|^{-1} \right\},$$

where g' is the derivative of g .

The proofs of the following two lemmas are straightforward and will be omitted.

1.6. Lemma. *Let (a_i, b_i) , $i = 1, 2, \dots, n$, be disjoint open intervals, where $n \leq \infty$ and $-\infty \leq a_i < b_i \leq \infty$, and let $\varepsilon > 0$. For each $i = 1, 2, \dots$, we are*

given a smooth Lipschitz homeomorphism $g_i: \mathbf{R} \rightarrow \mathbf{R}$ such that $g_i|_{\mathbf{R} \setminus (a_i, b_i)} = \text{id}$ and $\text{bilip}(g_i) \leq 1 + \varepsilon$. Define $g: \mathbf{R} \rightarrow \mathbf{R}$ by

$$g(x) = \begin{cases} g_i(x) & \text{if } a_i < x < b_i, \ i = 1, 2, \dots, \\ -x & \text{otherwise.} \end{cases}$$

Then g is a smooth Lipschitz homeomorphism with $\text{bilip}(g) \leq 1 + \varepsilon$.

1.7. Lemma. Let (a_i, b_i) , $i = 1, 2, \dots < n$, be disjoint open intervals in $[0, \infty)$ where $n \leq \infty$ and $0 \leq a_i < b_i \leq \infty$ and let $\varepsilon > 0$. For each $i = 1, 2, \dots$, we are given a smooth Lipschitz homeomorphism $g_i: \mathbf{R} \rightarrow \mathbf{R}$ such that $g_i|_{\mathbf{R} \setminus (-b_i, -a_i) \cup (a_i, b_i)} = \text{id}$ (i.e., $g_i(x) = -x$ if $|x| \notin (a_i, b_i)$) and $\text{bilip}(g_i) \leq 1 + \varepsilon$. Define $g: \mathbf{R} \rightarrow \mathbf{R}$ by

$$g(x) = \begin{cases} g_i(x) & \text{if } a_i < |x| < b_i, \ i = 1, 2, \dots, \\ -x & \text{otherwise.} \end{cases}$$

Then g is a smooth Lipschitz homeomorphism with $\text{bilip}(g) \leq 1 + \varepsilon$.

1.8. Lemma. Let $h \in H([a, b])$, $-\infty \leq a < b \leq \infty$ and $\varepsilon > 0$. Suppose $h(x) \neq x$ if and only if $x \neq a, b$. Then there is an $f \in H([a, b])$ and a smooth Lipschitz homeomorphism $g: \mathbf{R} \rightarrow \mathbf{R}$ such that $f(a) = a$, $f(b) = b$, $g|_{\mathbf{R} \setminus (a, b)} = \text{id}$, $\text{bilip}(g) \leq 1 + \varepsilon$ and $g|_{[a, b]} = fhf^{-1}$.

Proof. We may assume that $h(x) > x$ for all $x \in (a, b)$ (otherwise replace h by h^{-1} in the argument). Let $k: \mathbf{R} \rightarrow \mathbf{R}$ be a nonnegative smooth map such that $k^{-1}(0) = \mathbf{R} \setminus (a, b)$ and let

$$\delta = \frac{\varepsilon}{(1 + \varepsilon) \cdot \sup_{x \in \mathbf{R}} |k'(x)|} > 0.$$

Define a smooth map $g: \mathbf{R} \rightarrow \mathbf{R}$ by $g(x) = x + \delta k(x)$. Since $g'(x) = 1 + \delta k'(x)$, we have

$$\frac{1}{1 + \varepsilon} = 1 - \frac{\varepsilon}{1 + \varepsilon} \leq 1 - \delta |k'(x)| \leq |g'(x)| \leq 1 + \delta |k'(x)| < 1 + \varepsilon.$$

Thus g is a smooth Lipschitz homeomorphism with $\text{bilip}(g) \leq 1 + \varepsilon$ by Lemma 1.5. Note that $g(x) > x$ for all $x \in (a, b)$. We shall show that h is conjugate to $g|_{[a, b]}$. To this end, let $c \in (a, b)$. We can easily verify that $h^n(c) \rightarrow a$ and $g^n(c) \rightarrow a$ as $n \rightarrow -\infty$, and also $h^n(c) \rightarrow b$ and $g^n(c) \rightarrow b$ as $n \rightarrow \infty$. Let $f_0: [c, h(c)] \rightarrow [c, g(c)]$ be a homeomorphism such that $f_0(c) = c$ and $f_0(h(c)) = g(c)$. Extend f_0 to a homeomorphism $f: [a, b] \rightarrow [a, b]$ by $f(a) = a$, $f(b) = b$ and

$$f[h^n(c), h^{n+1}(c)] = g^n f_0 h^{-n} \quad \text{for each } n \in \mathbf{Z}.$$

Then $fh = gf$; hence h is conjugate to $g|_{[a, b]}$. \square

1.9. Lemma. Let $h \in H([-b, -a] \cup [a, b])$, $0 \leq a < b \leq \infty$ and $\varepsilon > 0$. Suppose $h(\pm a) = \mp a$, $h(\pm b) = \mp b$ and h has no point of period 2 except $\pm a$ and

$\pm b$. Then there is an $f \in H([-b, -a] \cup [a, b])$ and a smooth Lipschitz homeomorphism $g: \mathbf{R} \rightarrow \mathbf{R}$ such that $f(\pm a) = \pm a$, $f(\pm b) = \pm b$, $g|_{\mathbf{R} \setminus ((-b, -a) \cup (a, b))} = r$ (the reflection), $\text{bilip}(g) \leq 1 + \varepsilon$ and $g|_{[-b, -a] \cup [a, b]} = fhf^{-1}$.

Proof. We may assume that $h^2(x) > x$ for all $x \in (a, b)$ (otherwise replace h^2 by h^{-2} in the argument). It follows that $h^2(x) < x$ for all $x \in (-b, -a)$. Let $k: \mathbf{R} \rightarrow \mathbf{R}$ and $\delta > 0$ be as in the proof of Lemma 1.8. Define a smooth map $g: \mathbf{R} \rightarrow \mathbf{R}$ by

$$g(x) = \begin{cases} -x - \delta k(x) & \text{for } x \geq 0, \\ -x + \delta k(-x) & \text{for } x \leq 0. \end{cases}$$

Since $g'(x) = -1 - \delta k'(|x|)$, we have

$$\frac{1}{1 + \varepsilon} = 1 - \frac{\varepsilon}{1 + \varepsilon} \leq 1 - \delta |k'(x)| \leq |g'(x)| \leq 1 + \delta |k'(x)| < 1 + \varepsilon.$$

Then g is a smooth Lipschitz homeomorphism with $\text{bilip}(g) \leq 1 + \varepsilon$ by Lemma 1.5. Note that $g(\pm a) = \mp a$, $g(\pm b) = \mp b$ and $g^2(x) > x$ for all $x \in (a, b)$ (hence $g^2(x) < x$ for all $x \in (-b, -a)$).

We want to show that h is conjugate to $g|_{[-b, -a] \cup [a, b]}$. To this end, let $c \in (a, b)$. Observe that $h^{2n}(c) \rightarrow a$ and $g^{2n}(c) \rightarrow a$ as $n \rightarrow -\infty$; $h^{2n}(c) \rightarrow b$ and $g^{2n}(c) \rightarrow b$ as $n \rightarrow \infty$; $h^{2n+1}(c) \rightarrow -a$ and $g^{2n+1}(c) \rightarrow -a$ as $n \rightarrow -\infty$; and $h^{2n+1}(c) \rightarrow -b$ and $g^{2n+1}(c) \rightarrow -b$ as $n \rightarrow \infty$. Let $f_0: [c, h^2(c)] \rightarrow [c, g^2(c)]$ be a homeomorphism such that $f_0(c) = c$ and $f_0(h^2(c)) = g^2(c)$. Extend f_0 to a homeomorphism $f: [-b, -a] \cup [a, b] \rightarrow [-b, -a] \cup [a, b]$ by $f(\pm a) = \pm a$, $f(\pm b) = \pm b$ and

$$f|[h^n(c), h^{n+2}(c)] = g^n f_0 h^{-n} \quad \text{for each } n \in \mathbf{Z}.$$

Then $fh = gf$; hence h is conjugate to $g|_{[-b, -a] \cup [a, b]}$. \square

Proof of Theorem 1.1. Let $h \in H(\mathbf{R})$ and $\varepsilon > 0$. First we assume that h is increasing (i.e., order preserving). Let $\text{Fix } h$ denote the fixed point set of h . The case $\text{Fix } h = \emptyset$ is Proposition 1.2. Note that $\mathbf{R} \setminus \text{Fix } h$ is the disjoint union of open intervals (a_i, b_i) , $i = 1, \dots, n$, where $n \leq \infty$ and $-\infty \leq a_i < b_i \leq \infty$. For each i , let $h_i = h|_{[a_i, b_i]} \in H([a_i, b_i])$ and $\varepsilon_i = \varepsilon$ for each i . By Lemma 1.8, we have a smooth Lipschitz homeomorphism $g_i: \mathbf{R} \rightarrow \mathbf{R}$ and $f_i \in H([a_i, b_i])$ such that $g_i|_{\mathbf{R} \setminus (a_i, b_i)} = \text{id}$, $\text{bilip}(g_i) \leq 1 + \varepsilon_i$, $f_i(a_i) = a_i$, $f_i(b_i) = b_i$ and $g_i|[a_i, b_i] = f_i h_i f_i^{-1}$. Define $f, g: \mathbf{R} \rightarrow \mathbf{R}$ by $f|_{\text{Fix } h} = g|_{\text{Fix } h} = \text{id}$, $f|[a_i, b_i] = f_i$ and $g|[a_i, b_i] = g_i$ for each i . Then f is clearly a homeomorphism and g is a smooth Lipschitz homeomorphism by Lemma 1.6. Since $g = fhf^{-1}$, h is conjugate to g .

Next we assume that h is decreasing (i.e., order reversing). By Lemma 1.4, we may assume that $h(0) = 0$, $h(x) = -x$ if x is a periodic point of h . Let $\text{Per } h$ denote the periodic point set of h . The case $\text{Per } h = \mathbf{R}$ is Proposition 1.3. Note that $[0, \infty) \setminus \text{Per } h$ is the disjoint union of open intervals (a_i, b_i) ,

$i = 1, 2, \dots, n$, where $n \leq \infty$ and $0 \leq a_i < b_i \leq \infty$, and $(-\infty, 0] \setminus \text{Per } h$ is the disjoint union of

$$(-b_i, -a_i) = h((a_i, b_i)), \quad i = 1, 2, \dots, n.$$

Let $h_i = h|[-b_i, -a_i] \cup [a_i, b_i] \in H([-b_i, -a_i] \cup [a_i, b_i])$ and $\varepsilon_i = \varepsilon$ for each i . By Lemma 1.9, we have a smooth Lipschitz homeomorphism $g_i: \mathbf{R} \rightarrow \mathbf{R}$ and $f_i \in H([-b_i, -a_i] \cup [a_i, b_i])$ such that $g_i|_{\mathbf{R} \setminus ((-b_i, -a_i) \cup (a_i, b_i))} = \text{id}$, $\text{bilip}(g_i) \leq 1 + \varepsilon_i$, $f_i(\pm a_i) = \pm a_i$, $f_i(\pm b_i) = \pm b_i$ and $g_i|[-b_i, -a_i] \cup [a_i, b_i] = f_i h_i f_i^{-1}$. Define $f, g: \mathbf{R} \rightarrow \mathbf{R}$ by $f|_{\text{Per } h} = \text{id}$, $g|_{\text{Per } h} = r$, $f|[-b_i, -a_i] \cup [a_i, b_i] = f_i$ and $g|[-b_i, -a_i] \cup [a_i, b_i] = g_i$ for each i . Then f is clearly a homeomorphism and g is a smooth Lipschitz homeomorphism by Lemma 1.7. Since $g = fhf^{-1}$, h is conjugate to g . \square

Remark. One should remark that Theorem 1.1 does not hold for $(0, \infty)$. In fact, any decreasing (order reversing) homeomorphism of $(0, \infty)$ (e.g., $x \rightarrow 1/x$) is not conjugate to any uniform homeomorphism.

2. THE CONJUGATION PROBLEM (A): COMPLETE METRIC CASE

For a given metric space $X = (X, d)$, let d_1 be the metric on X defined by $d_1(x, y) = \min\{d(x, y), 1\}$. Then d_1 is uniformly equivalent to d . Replacing d by d_1 , we may assume without loss of generality that the metric d of X is bounded by 1. Let $C(X)$ denote the space of maps of X into itself with the sup-metric d^* defined by

$$d^*(f, g) = \sup\{d(f(x), g(x)) | x \in X\}.$$

Let $C_U(X)$ denote the subset consisting of uniformly continuous maps. Then $C_U(X)$ is closed in $C(X)$; hence $\overline{H_U(X)} \subset C_U(X)$ where $\overline{H_U(X)}$ is the closure of $H_U(X)$ in $C(X)$. Our main result in this section is the following theorem.

2.1. Theorem. *Let (X, d) be a complete metric space. Then each $h \in H(X) \cap \overline{H_U(X)}$ is conjugate to a $g \in H_U(X)$.*

Proof. For each $i \in \mathbf{Z}$ ($=$ integers), let X_i denote a copy of X and let $X^{\mathbf{Z}} = \prod_{i \in \mathbf{Z}} X_i$ with the product metric ρ defined by

$$\rho((x_i)_{i \in \mathbf{Z}}, (y_i)_{i \in \mathbf{Z}}) = \sup\{2^{-|i|} \cdot d(x_i, y_i) | i \in \mathbf{Z}\}.$$

Let $\pi_i: X^{\mathbf{Z}} \rightarrow X_i = X$, $i \in \mathbf{Z}$, be the projections and $S: X^{\mathbf{Z}} \rightarrow X^{\mathbf{Z}}$ be the left-shift homeomorphism of $X^{\mathbf{Z}}$ defined by $\pi_i \circ S = \pi_{i+1}$, that is,

$$S(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) = (\dots, x_{-1}, x_0, x_1, x_2, x_3, \dots).$$

Then $S \in H_U(X^{\mathbf{Z}})$. We define the infinite graph of h as follows:

$$X_h = \{(x_i)_{i \in \mathbf{Z}} \in X^{\mathbf{Z}} | x_n = h^n(x_0) \text{ for } n \neq 0\}.$$

Then clearly $S(X_h) = X_h$ and $\pi = \pi_0|_{X_h}: X_h \rightarrow X$ is a homeomorphism of X_h onto X . Thus we have a commutative diagram:

$$\begin{array}{ccc} X_h & \xrightarrow{S} & X_h \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{h} & X \end{array}$$

Let $X_\infty = \varprojlim (X_i, h_i) \subset \prod_{i=0}^\infty X_i$ denote the inverse limit of $h_i = h: X_i \rightarrow X_{i-1}$, $i = 1, 2, \dots$, that is,

$$\begin{aligned} X_\infty &= \{(x_0, x_1, \dots) | h(x_n) = x_{n-1} \text{ for each } n\} \\ &= \{(x_0, x_1, \dots) | x_n = h^{-n}(x_0) \text{ for } n > 0\}. \end{aligned}$$

The metric of X_∞ is the product metric defined similarly as the metric ρ of X^Z . Define $q: X_h \rightarrow X_\infty$ by

$$q(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) = (x_0, x_{-1}, x_{-2}, \dots).$$

Obviously q is uniformly continuous. Since the size of X_n is uniformly small for large n (under the product metric), uniform continuity of each h^n implies uniform continuity of q^{-1} . Hence q is a uniform homeomorphism.

We want to show that X_∞ is uniformly homeomorphic to X . By hypothesis, each bonding map $h_i: X_i \rightarrow X_{i-1}$ can be approximated (in the sup-metric d^* of $C(X)$) by uniform homeomorphisms. In the following construction, we employ a procedure of M. Brown [Br] (cf. [Cu₁, Theorem 3.1]) showing that two inverse limits of compacta are homeomorphic:

Given $\varepsilon_0 = 2^{-1}$, we can choose a uniform homeomorphism $g_1: X_1 \rightarrow X_0$ and $0 < \varepsilon_1 < 2^{-2}$ so that $d^*(g_1, h_1) < \varepsilon_0$ and for all $x, y \in X_1$, $d(x, y) < 2\varepsilon_1$ implies $d(h_1(x), h_1(y)) < \varepsilon_0$ and $d(g_1(x), g_1(y)) < \varepsilon_0$. Inductively given uniform homeomorphisms $g_i: X_i \rightarrow X_{i-1}$ and $0 < \varepsilon_i < 2^{-(i+1)}$ for $i = 1, \dots, n-1$, we can choose a uniform homeomorphism $g_n: X_n \rightarrow X_{n-1}$ and $0 < \varepsilon_n < 2^{-(n+1)}$ so that $d^*(g_n, h_n) < \varepsilon_{n-1}$ and for all $x, y \in X_n$, $d(x, y) < 2\varepsilon_n$ implies $d(h_{in}(x), h_{in}(y)) < \varepsilon_{n-1}$ and $d(g_{in}(x), g_{in}(y)) < \varepsilon_{n-1}$ for each $i = 0, 1, \dots, n-1$, where $h_{in} = h_{i+1} \circ h_{i+1n}$ and $g_{in} = g_{i+1} \circ g_{i+1n}$ ($h_{nn} = g_{nn} = \text{id}$). Thus we obtain uniform homeomorphisms $g_n: X_n \rightarrow X_{n-1}$, $n = 1, 2, \dots$. By the choice of ε_n , it is straightforward to verify that for all $0 \leq i < j < k$,

$$(*) \quad d^*(g_{ij}h_{jk}, g_{ik}) < 2^{-j} \quad \text{and} \quad d^*(h_{ij}g_{jk}, h_{ik}) < 2^{-j}.$$

First we will show that X_∞ is uniformly homeomorphic to

$$X'_\infty = \varprojlim (X_i, g_i).$$

Let $h_{i\infty}: X_\infty \rightarrow X_i$ and $g_{i\infty}: X'_\infty \rightarrow X_i$ be projections. From (*), $\{g_{ij}h_{j\infty}\}_{j>1}$ is a uniform Cauchy sequence of uniformly continuous maps from X_∞ to X_i .

Then we can define a uniformly continuous map $\varphi: X_\infty \rightarrow X'_\infty$ by $g_{i\infty}\varphi = \lim_{j \rightarrow \infty} g_{ij}h_{j\infty}$. Likewise we have a uniformly continuous map $\psi: X'_\infty \rightarrow X_\infty$ defined by $h_{i\infty}\psi = \lim_{j \rightarrow \infty} h_{ij}g_{j\infty}$. From (*), it follows that

$$h_{i\infty}\psi\varphi = \lim_{\substack{j,k \rightarrow \infty \\ i < j < k}} h_{ij}g_{jk}h_{k\infty} = h_{i\infty};$$

hence $\psi\varphi = \text{id}$. Similarly $\varphi\psi = \text{id}$. Therefore φ is a uniform homeomorphism. Now $g_{0\infty}: X'_\infty \rightarrow X_0 = X$ is clearly a homeomorphism which is uniformly continuous. Since each g_i^{-1} is uniformly continuous, so is $g_{0\infty}^{-1}$. Hence $g_{0\infty}$ is also a uniform homeomorphism.

Thus we obtain a uniform homeomorphism $f = g_{0\infty}\varphi q: X_h \rightarrow X$. Then h is conjugate to $g = fSf^{-1} \in H_U(X)$ since $g = f\pi^{-1}h\pi f^{-1}$. \square

Let (X, d) be any metric space (complete or not) and let $h \in H(X)$. In the above proof, we constructed the infinite graph (X_h, ρ) of h and the homeomorphism $\pi: X_h \rightarrow X$ such that $S = \pi^{-1}h\pi$ is the left-shift homeomorphism. Define a metric d_h on X by $d_h(x, y) = \rho(\pi^{-1}(x), \pi^{-1}(y))$. Then d_h is equivalent to d . Since S is in fact a Lipschitz homeomorphism, we have proved

2.2. Corollary. *Given any homeomorphism h of a metric space X onto itself, there is an equivalent metric on X making h a Lipschitz homeomorphism.*

Let (X, d) be any noncomplete metric space and let $h \in H(X) \cap \overline{H_U(X)}$. Since h is uniformly continuous, h extends to a map $\tilde{h}: \tilde{X} \rightarrow \tilde{X}$ of the completion of (X, d) . Then obviously $\tilde{h} \in H_U(\tilde{X})$. Let \overline{X}_h be the closure of the infinite graph X_h of h in $\tilde{X}^{\mathbb{Z}} = \prod_{i \in \mathbb{Z}} \tilde{X}_i$. It is easy to see

$$\overline{X}_h = \{(x_i)_{i \in \mathbb{Z}} \in \tilde{X}^{\mathbb{Z}} \mid x_i = \tilde{h}(x_{i-1}) \text{ for all } i \in \mathbb{Z}\}.$$

By the same arguments as in the proof of Theorem 2.1, we have a uniform homeomorphism $\tilde{f}: \overline{X}_h \rightarrow \tilde{X}$. In general, we do not know whether the pair (\overline{X}_h, X_h) is uniformly homeomorphic to (\tilde{X}, X) . In case X is the interior of a complete metric n -manifold \tilde{X} , $f = \tilde{f}|_{X_h}: X_h \rightarrow X$ is a uniform homeomorphism by the invariance of domain. (In fact, X_h is the interior of the manifold \overline{X}_h by Lemma 3.2(1).) Thus we have

2.3. Corollary. *Let X be the interior of a complete metric n -manifold \tilde{X} . Then each $h \in H(X) \cap \overline{H_U(X)}$ is conjugate to a $g \in H_U(X)$ (which extends to a uniform homeomorphism of \tilde{X}).*

We conclude this section by exhibiting an example $h \in H(X) \cap C_U(X)$ for a connected complete metric 2-manifold $X \subset \mathbb{R}^2$ with boundary $\partial X \neq \emptyset$ such that h is not conjugate to any uniform homeomorphism of X .

2.4. Example. Define a_n , $n \in \mathbb{Z}$, as follows: $a_n = 1$ for $n < 0$, $a_0 = 2$ and $a_n = (a_{n-1} + 1)^2$ for $n > 0$. For each $n \in \mathbb{Z}$, we denote the rectangular 2-cell $[2n, 2n+1] \times [0, a_n]$ by D_n and its boundary and interior by ∂D_n and \dot{D}_n , respectively. Let $X = \mathbb{R}^2 \setminus \bigcup_{n \in \mathbb{Z}} \dot{D}_n$. Then X is a complete metric connected 2-manifold with boundary $\partial X = \bigcup_{n \in \mathbb{Z}} \partial D_n$. Let $h: X \rightarrow X$ be the canonical p.l. homeomorphism which carries each ∂D_n to ∂D_{n-1} . Then $h \in H(X) \cap C_U(X)$. For each $f \in H(X)$ and each $n \in \mathbb{Z}$, we denote $f(\partial D_n) = \partial D_{f_*(n)}$. Then $h_*(n) = n - 1$.

We want to show that h is not conjugate to any $g \in H_U(X)$. So suppose the contrary, that h is conjugate to a $g \in H_U(X)$. Then as is easily observed, $g_*(n) \neq n$ for any $n \in \mathbb{Z}$. From uniform continuity of g and g^{-1} , there is an integer $m > 0$ such that for $A \subset X$, $\text{diam } A \leq 1/m$ implies $\text{diam } g(A) \leq 1$ and $\text{diam } g^{-1}(A) \leq 1$. Choose an integer $n > 0$ so that $a_{n-1} \geq 2m$. Then $g_*(n) < n$ or $g_*(n) > n$.

Case 1: $g_*(n) < n$. Decompose $\partial D_{g_*(n)}$ into $2m(a_{g_*(n)} + 1)$ segments A_i with length $= 1/m$, that is, $\partial D_{g_*(n)} = \bigcup \{A_i | i = 1, \dots, 2m(a_{g_*(n)} + 1)\}$. Then

$$\begin{aligned} \text{diam } g^{-1}(\partial D_{g_*(n)}) &\leq \sum \{\text{diam } g^{-1}(A_i) | i = 1, \dots, 2m(a_{g_*(n)} + 1)\} \\ &\leq 2m(a_{g_*(n)} + 1) < (a_{n-1} + 1)^2 = a_n \\ &< \text{diam } \partial D_n = \text{diam } g^{-1}(\partial D_{g_*(n)}). \end{aligned}$$

This is a contradiction.

Case 2: $g_*(n) > n$. We can similarly write

$$\partial D_n = \bigcup \{A_i | i = 1, \dots, 2m(a_n + 1)\},$$

where each A_i is a segment with length $= 1/m$. Then

$$\begin{aligned} \text{diam } g(\partial D_n) &\leq \sum \{\text{diam } g(A_i) | i = 1, \dots, 2m(a_n + 1)\} \\ &\leq 2m(a_n + 1) < (a_n + 1)^2 = a_{n+1} \leq a_{g_*(n)} \\ &< \text{diam } \partial D_{g_*(n)} = \text{diam } g(\partial D_n). \end{aligned}$$

This is also a contradiction. \square

3. THE CONJUGATION PROBLEM (B): MANIFOLD CASE

Let X and Y be spaces and \mathcal{U} an open cover of Y . We say that maps $f, g: X \rightarrow Y$ are \mathcal{U} -close if for each x there is a $U \in \mathcal{U}$ such that

$$\{f(x), g(x)\} \subset U.$$

Maps $f, g: X \rightarrow Y$ are \mathcal{U} -homotopic if there is a homotopy $h: X \times I \rightarrow Y$ such that $h_0 = f$, $h_1 = g$ and each $h(\{x\} \times I)$ is contained in some member

of \mathcal{U} . A map $f: X \rightarrow Y$ is called a *near homeomorphism* if for each open cover \mathcal{U} of Y there is a homeomorphism $g: X \rightarrow Y$ which is \mathcal{U} -close to f . A map $f: X \rightarrow Y$ is called a *fine homotopy equivalence* if for each open cover \mathcal{U} of Y there is a map $g: Y \rightarrow X$ such that fg is \mathcal{U} -homotopic to id and gf is $f^{-1}(\mathcal{U})$ -homotopic to id . The proof of the following lemma is rather trivial and will be omitted.

3.1. Lemma. *Let $f: X \rightarrow Y$ be a fine homotopy equivalence. Then a map $g: Y \rightarrow Z$ is a fine homotopy equivalence if and only if $gf: X \rightarrow Z$ is a fine homotopy equivalence.*

We say that a subset Y of a metric space $X = (X, d)$ is *map-dense* in X if for any compactum $A \subset X$ and any $\varepsilon > 0$, there is a map $f: A \rightarrow Y$ with $d^*(f, \text{id}) < \varepsilon$ [Cu₂]. A closed subset K of X is called a *Z-set* in X if for any open cover \mathcal{U} of X there is a map $f: X \rightarrow X \setminus K$ with is \mathcal{U} -close to id (cf. [Ch₂]). In case X is an ANR, this is equivalent to the condition: $X \setminus K$ is map-dense in X .

Throughout this section, we will use the following convention. Given a metric space $X = (X, d)$, let \tilde{X} denote the completion of X and $X^* = \tilde{X} \setminus X$. To simplify notation, the metric of \tilde{X} will also be denoted by d . As mentioned before, we may assume that d is bounded by 1. For each $i \in \mathbb{Z}$, let \tilde{X}_i denote a copy of \tilde{X} and let $\tilde{X}^{\mathbb{Z}} = \prod_{i \in \mathbb{Z}} \tilde{X}_i$ with the product metric ρ defined as in the proof of Theorem 2.1. Let $\pi_i: \tilde{X}^{\mathbb{Z}} \rightarrow \tilde{X}_i = \tilde{X}$, $i \in \mathbb{Z}$, denote the projections and $S: \tilde{X}^{\mathbb{Z}} \rightarrow \tilde{X}^{\mathbb{Z}}$ be the left-shift homeomorphism defined by $\pi_i S = \pi_{i+1}$. Let X_h be the infinite graph of $h \in H(X)$ in $\tilde{X}^{\mathbb{Z}}$, that is,

$$X_h = \{(x_i)_{i \in \mathbb{Z}} \in \tilde{X}^{\mathbb{Z}} \mid x_0 \in X, x_n \in h^n(x_0) \text{ for } n \neq 0\},$$

\overline{X}_h the closure of X_h in $\tilde{X}^{\mathbb{Z}}$ and $X_h^* = \overline{X}_h \setminus X_h$. Then $S(X_h) = X_h$, $S(\overline{X}_h) = \overline{X}_h$, $S(X_h^*) = X_h^*$, $\pi_0(X_h) = X$, $\pi_0(\overline{X}_h) = \tilde{X}$, $\pi_0(X_h^*) = X^*$ and $\pi = \pi_0|_{X_h}: X_h \rightarrow X$ is a homeomorphism. We have the following commutative diagram:

$$\begin{array}{ccc} \overline{X}_h & \xrightarrow{S} & \overline{X}_h \\ \cup & & \cup \\ X_h & \xrightarrow{S} & X_h \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{h} & X \end{array}$$

Thus the conjugation problem (B) is equivalent to the problem of whether the pair (\overline{X}_h, X_h) is homeomorphic (\cong) to (\tilde{X}, X) .

3.2. Lemma. *In the above diagram, suppose that h extends to a map $\tilde{h}: \tilde{X} \rightarrow \tilde{X}$. Then (1) if X is map-dense in \tilde{X} , then X_h is also map-dense in \overline{X}_h ,*

(2) if \tilde{h} is proper, then so is $\tilde{\pi} = \pi_0|_{\overline{X}_h}: \overline{X}_h \rightarrow \tilde{X}$, and (3) if \tilde{h} is a near homeomorphism, then $\tilde{\pi}$ is a near homeomorphism.

Proof. First observe

$$\overline{X}_h = \{(x_i)_{i \in \mathbb{Z}} \in \tilde{X}^{\mathbb{Z}} | x_i = \tilde{h}(x_{i-1}) \text{ for all } i \in \mathbb{Z}\}.$$

Let $\tilde{X}_\infty = \varprojlim (\tilde{X}_i, \tilde{h}_i)$, where $\tilde{h}_i = \tilde{h}: \tilde{X}_i \rightarrow \tilde{X}_{i-1}$. Then we have a homeomorphism $q: \overline{X}_h \rightarrow \tilde{X}_\infty$ defined by

$$q(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) = (x_0, x_{-1}, x_{-2}, \dots).$$

If \tilde{h} is a near homeomorphism, then the projection $\tilde{h}_{0\infty}: \tilde{X}_\infty \rightarrow \tilde{X}_0 = \tilde{X}$ is a near homeomorphism by [Cu₁, Theorem 3.3]. (This is proved by the same arguments as Theorem 2.1 using open covers \mathcal{U}_i of \tilde{X}_i instead of $\varepsilon_i > 0$ where \mathcal{U}_0 can be chosen arbitrarily small.) Hence $\tilde{\pi} = \tilde{h}_{0\infty}q: \overline{X}_h \rightarrow \tilde{X}_0 = \tilde{X}$ is a near homeomorphism. If \tilde{h} is proper, then $\tilde{h}_{0\infty}: \tilde{X}_\infty \rightarrow \tilde{X}$ is also proper because each bonding map \tilde{h}_i is proper. Hence $\tilde{\pi} = \tilde{h}_{0\infty}q$ is proper. Thus we have (3) and (2).

Finally we will prove (1). Let A be a compactum in \overline{X}_h and $\varepsilon > 0$. Choose an integer $n > 0$ so that $2^{-n} < \varepsilon$. Since $\pi_{-n}(A)$ is compact, there is a $\delta > 0$ such that if $a \in \pi_{-n}(A)$, $x \in \tilde{X}$ and $d(a, x) < \delta$, then $d(h^i(a), h^i(x)) < \varepsilon$ for $i = 0, 1, \dots, 2n$. Since X is map-dense in \tilde{X} , there is a map $f: \pi_{-n}(A) \rightarrow X$ with $d^*(f, \text{id}) < \delta$. Since $\pi_{-n}|_{X_h}: X_h \rightarrow X$ is a homeomorphism, we have a map $g = (\pi_{-n}|_{X_h})^{-1} \circ f \circ (\pi_{-n}|_A): A \rightarrow X_h$. For each $x = (x_i)_{i \in \mathbb{Z}} \in A$ let $g(x) = (y_i)_{i \in \mathbb{Z}}$. Then $y_i = h^{i+n}f(x_{-n})$ and $x_i = h^{i+n}(x_{-n})$ for all $i \in \mathbb{Z}$. Since $d(f(x_{-n}), x_{-n}) < \delta$ and $x_{-n} \in \pi_{-n}(A)$,

$$2^{-|i|}d(y_i, x_i) \leq d(h^{i+n}f(x_{-n}), h^{i+n}(x_{-n})) < \varepsilon$$

provided $|i| \leq n$. Finally if $|i| > n$ then $2^{-|i|}d(y_i, x_i) < 2^{-n} < \varepsilon$. Therefore $\rho(g(x), x) < \varepsilon$ for each $x \in A$. So X_h is map-dense in \overline{X}_h . \square

Now we give a version of Corollary 2.3.

3.3. Theorem. *Let $n \neq 4, 5$ and X be the interior of an n -manifold \tilde{X} . If $h \in H(X)$ extends to a proper map \tilde{h} of \tilde{X} , then h is conjugate to a $g \in H(X)$ which extends to $\tilde{g} \in H(\tilde{X})$.*

Proof. Since the boundary $\partial\tilde{X} = X^*$ of \tilde{X} has a collar in \tilde{X} , the inclusion $X \subset \tilde{X}$ is a fine homotopy equivalence. By Lemma 3.1, $\tilde{h}: \tilde{X} \rightarrow \tilde{X}$ is also a fine homotopy equivalence, hence a CE-map. By the result of Siebenmann [Si], \tilde{h} is a near homeomorphism. Thus we have a homeomorphism $f: \overline{X}_h \rightarrow \tilde{X}$ by Lemma 3.2. By the invariance of domain, $f(X_h) = X$ since X_h is the interior of \overline{X}_h by Lemma 3.2. Hence $(\overline{X}_h, X_h) \cong (\tilde{X}, X)$, which completes the proof. \square

As seen in Example 2.4, we cannot require $g \in H_U(X)$ in the above theorem even if h extends to $\tilde{h} \in H(\tilde{X}) \cap C_U(\tilde{X})$.

In the following example, we show that, in general, the hypothesis that h extends to a proper map of \tilde{X} cannot be eliminated from the above theorem. Let B^n and S^{n-1} denote the closed unit n -ball and the unit $(n-1)$ -sphere of \mathbf{R}^n , i.e., $B^n = \{x \in \mathbf{R}^n \mid \|x\| \leq 1\}$ and $S^{n-1} = \{x \in \mathbf{R}^n \mid \|x\| = 1\}$. Let $\dot{B}^n = B^n \setminus S^{n-1}$ be the interior of B^n .

3.4. Example. For each $n \geq 2$, there is an $h \in H(\dot{B}^n)$ such that h is not conjugate to any uniform homeomorphism. Moreover, h can be chosen ε -close to id for any given $\varepsilon > 0$.

Construction. First we consider the case $n = 2$ without the ε condition. For each $i = 1, 2, \dots$, let $S_i^1 = \{x \in \mathbf{R}^2 \mid \|x\| = i/i + 1\}$. Let $\alpha_2: \dot{B}^2 \rightarrow \dot{B}^2$ be a homeomorphism such that for each i , $\alpha_2|_{S_i^1} = \text{id}$ and α_2 performs a Dehn twist (in any direction) in the annulus region between S_i^1 and S_{i+1}^1 :

For $n \geq 3$, we define $\alpha_n: \dot{B}^n \rightarrow \dot{B}^n$ by suspending α_2 as follows:

$$\alpha_n(x) = \begin{cases} x & \text{if } (x_1, x_2) = (0, 0), \\ \left(\frac{\|(x_1, x_2)\|}{\|x\|} \cdot \alpha_2 \left(\frac{\|x\|}{\|(x_1, x_2)\|} \cdot (x_1, x_2) \right), x_3, \dots, x_n \right) & \text{otherwise.} \end{cases}$$

Let $h = \alpha_n \in H(\dot{B}^n)$. We claim that h is not conjugate to any uniform homeomorphism. Suppose on the contrary that h is conjugate to some $g \in H_U(\dot{B}^n)$. We write $g = fhf^{-1}$ for some $f \in H(\dot{B}^n)$. Since $g \in H_U(\dot{B}^n)$, g extends to a $\tilde{g} \in H(B^n)$. Note $g|_{\bigcup_{i=1}^\infty f(S_i^{n-1})} = \text{id}$ since $h|_{\bigcup_{i=1}^\infty S_i^{n-1}} = \text{id}$, where $S_i^{n-1} = \{x \in \mathbf{R}^n \mid \|x\| = i/(i+1)\}$. It is easy to see that any neighborhood of each $x \in S^{n-1}$ intersects with $\bigcup_{i=1}^\infty f(S_i^{n-1})$. Therefore $\tilde{g}|_{S^{n-1}} = \text{id}$. Then there is a neighborhood V of S^{n-1} such that for each $x \in V$ the line segment between x and $\tilde{g}(x)$ misses the point $f(0)$.

Let A_i denote the closed annulus region between S_i^{n-1} and S_{i+1}^{n-1} , and let

$$R_i = \{(x_1, \dots, x_n) \in A_i \mid x_1 \geq 0 \text{ and } x_2 = 0\}$$

be an $(n-1)$ -cell in A_i and ∂R_i the boundary of R_i . Let $r: B^n \setminus \{0\} \rightarrow S^{n-1}$ be the radial retraction defined by $r(x) = x/\|x\|$. Then $k = (r|R_i)(rh|R_i)^{-1}: S^{n-1} \rightarrow S^{n-1}$ is a well-defined null-homotopic map. By choosing i large enough, $f(A_i) \subset V$ so $f(R_i) \cup gf(R_i) \subset V$. Then $f|R_i$ is homotopic to $gf|R_i = fh|R_i$ by the linear-path homotopy in $B^n \setminus \{f(0)\}$ fixing $f|\partial R_i = fh|\partial R_i$. Thus $r|R_i = rf^{-1}f|R_i$ is homotopic to $rh|R_i = rf^{-1}fh|R_i$ fixing $r|\partial R_i = rh|\partial R_i$. This homotopy induces a homotopy between $k = (r|R_i)(rh|R_i)^{-1}$ and $\text{id} = (rh|R_i)(rh|R_i)^{-1}: S^{n-1} \rightarrow S^{n-1}$. This is a contradiction.

To construct h ε -close to id , choose an integer $k > 2\pi/\varepsilon$. In the above construction, we replace the Dehn twist α_2 by a homeomorphism $\alpha'_2: \dot{B}^2 \rightarrow \dot{B}^2$

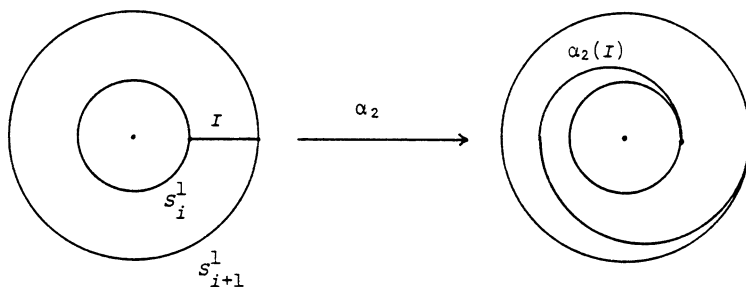


FIGURE 1

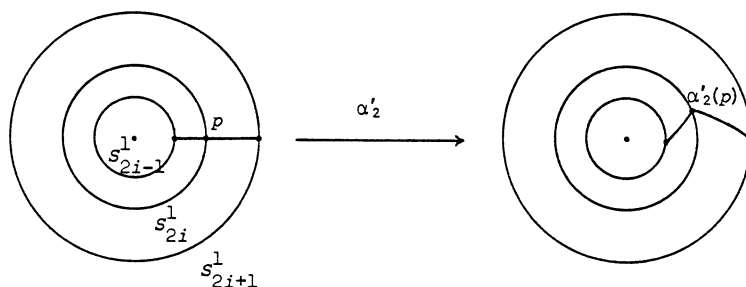


FIGURE 2

such that $\alpha'_2|_{S_{2i-1}^1} = \text{id}$, $\alpha'_2|_{S_{2i}^1}$ is a $2\pi/k$ -radian rotation. Similarly define $h = \alpha'_2: \dot{B}^n \rightarrow \dot{B}^n$ by suspending α'_2 . Then $d^*(h, \text{id}) < \varepsilon$. Similar argument as above shows that h^k is not conjugate to any uniform homeomorphism; hence h is not conjugate to any uniform homeomorphism.

As generalizations of (complete) n -manifolds, we consider Hilbert cube manifolds (Q -manifolds) and Hilbert manifolds (l_2 -manifolds). Since $l_2 \cong s[AB]$, l_2 -manifolds are also called s -manifolds. For a pair (M, N) of Q - (or l_2 -) manifolds, N is collared in M if and only if N is a Z -set in M [Ch₂, Theorem 16.2] or [Sa₁, footnote (1)]. Such a submanifold N is called a Z -submanifold of M and it can be considered as a “boundary” of M .

3.5. Theorem. *Let X be an open submanifold of a Q -manifold \tilde{X} with $X^* = \tilde{X} \setminus X$ a Z -submanifold. If $h \in H(X)$ extends to a proper map $\tilde{h}: \tilde{X} \rightarrow \tilde{X}$, then h is conjugate to a $g \in H(X)$ which extends to $\tilde{g} \in H(\tilde{X})$.*

Proof. As demonstrated in Theorem 3.3, $\tilde{h}: \tilde{X} \rightarrow \tilde{X}$ is a CE-map, hence a near homeomorphism by [Ch₂, Corollary 43.2]. Since \tilde{X} is locally compact, there is an open cover \mathcal{U} of \tilde{X} such that for any space Y if a map $g: Y \rightarrow \tilde{X}$ is \mathcal{U} -close to a proper map then g is also proper [Ch₂, Theorem 4.1, (2)]. From Lemma 3.2, $\pi: \bar{X}_h \rightarrow \tilde{X}$ is \mathcal{U} -homotopic to a homeomorphism $f: \bar{X}_h \rightarrow \tilde{X}$. Since π is proper, f is proper homotopic to π . Since X^* is a Z -set in \tilde{X} , $\pi^{-1}(X^*) = X_h^*$ is also a Z -set in \bar{X}_h by Lemma 3.2. Hence $f(X_h^*)$ is a Z -set in \tilde{X} .

The restriction $h^* = \tilde{h}|_{X^*}: X^* \rightarrow X^*$ is a CE-map, so again a near homeomorphism. The homeomorphism $q: \bar{X}_h \rightarrow \tilde{X}_\infty$ defined in the proof of Lemma

3.2 restricts to a homeomorphism $q^* = q|X_h^*: X_h^* \rightarrow X_\infty^* = \varprojlim (X_i^*, h_i^*)$, where $X_i^* = X^*$ and $h_i^* = \tilde{h}|X^*: X_i^* \rightarrow X_{i-1}^*$. By [Cu₁, Theorem 3.3], the projection $h_{0\infty}^*: X_\infty^* \rightarrow X_0^* = X^*$ is a near homeomorphism. Hence $\pi^* = \pi|X_h^* = h_{0\infty}^* q^*: X_h^* \rightarrow X^*$ is a near homeomorphism. Thus we have a homeomorphism $f^*: X_h^* \rightarrow X^*$ which is \mathcal{U} -homotopic to π^* , hence proper homotopic to $\pi^* = \pi|X_h^*$. Then f^* is proper homotopic to $f|X_h^*$. By the Homeomorphism Extension Theorem [Ch₂, Theorem 19.4], f^* extends to a homeomorphism $\tilde{f}: \bar{X}_h \rightarrow \tilde{X}$. Since $\tilde{f}(X_h^*) = X^*$, $\tilde{f}(X_h) = X$. Thus $(\bar{X}_h, X_h) \cong (\tilde{X}, X)$, which implies the existence of g required in the theorem. \square

Using the same procedure, we can prove a similar theorem for l_2 -manifolds. In this case, we do not need the assumption that \tilde{h} is proper, but instead we use the α -Approximation Theorem [Fe, Theorem 3.4] and the Homeomorphism Extension Theorem [AM].

3.6. Theorem. *Let X be an open submanifold of an l_2 -manifold \tilde{X} with $X^* = \tilde{X} \setminus X$ a Z -submanifold. If $h \in H(X)$ extends to a map $\tilde{h}: \tilde{X} \rightarrow \tilde{X}$, then h is conjugate to a $g \in H(X)$ which extends to $\tilde{g} \in H(\tilde{X})$. In particular, each $h \in H(X) \cap C_U(X)$ is conjugate to a $g \in H(X)$ which extends to $\tilde{g} \in H(\tilde{X})$. (Here the metric of X is inherited from the complete metric of \tilde{X} .)*

Let M be a Q -manifold or an l_2 -manifold. We say that a subset N of M is an (f-d) cap set for M if (1) $N = \bigcup_{n=1}^\infty N_n$, where each N_n is a (finite-dimensional) compact Z -set in M such that $N_n \subset N_{n+1}$, and (2) for each $\varepsilon > 0$, each integer $m > 0$, and each (finite-dimensional) compactum K in M , there is an integer $n \geq m$ and an embedding $f: K \rightarrow N_n$ such that $f|K \cap N_m = \text{id}$ and $d^*(f, \text{id}) < \varepsilon$ [Ch₁]. When we say that N is an (f-d) cap set for M , it means that N is a cap set (or an f-d cap set) for M . Let σ denote the set of all points in s having at most finitely many nonzero coordinates and let Σ denote the set of all points in s having at most finitely many coordinates not in $[-\frac{1}{2}, \frac{1}{2}]$. Then σ is an f-d cap set for s and Q , and Σ is a cap set for s and Q . Let l_2^f denote the linear span of the usual orthonormal basis of l_2 . Then l_2^f is an f-d cap set for l_2 ; hence $(l_2, l_2^f) \cong (s, \sigma)$ by the topological uniqueness of f-d cap sets [Ch₁, Theorem 6.2]. An f-d cap set for an l_2 - or Q -manifold is a σ -manifold and a cap set for an l_2 - or Q -manifold is a Σ -manifold [Ch₁]. An (f-d) cap set N for a Q -manifold M is considered as a “boundary” of M in some sense (see [Cu₂, Introduction]) and $M \setminus N$ is always an l_2 -manifold [Ch₁]. On the other hand, a cap set for a Q -manifold M is also considered as an “interior” of M in some sense (cf. [BP, Chapter V, Corollary 4.2]). We note that for an (f-d) cap set N for an l_2 - or Q -manifold M , the inclusions $N \subset M$ and $M \setminus N \subset M$ are fine homotopy equivalences. (This can be proved by using the Triangulation Theorems, the topological uniqueness of (f-d) cap sets and the fact that the inclusions $\sigma \subset \Sigma \subset s \subset Q$ are fine homotopy equivalences.)

3.7. Theorem. *Let X be an $(f-d)$ cap set for a Q -manifold \tilde{X} . If $h \in H(X)$ extends to a proper map $\tilde{h}: \tilde{X} \rightarrow \tilde{X}$, then h is conjugate to a $g \in H(X)$ which extends to $\tilde{g} \in H(\tilde{X})$.*

Proof. Similarly as in Theorem 3.5, $\overline{X}_h \cong \tilde{X}$. Since X is map-dense in \tilde{X} , X_h is also map-dense in \overline{X}_h by Lemma 3.2. Recall that $X_h \cong X$. By [Sa₂, Proposition 2.1] (the cap set version is similar), X_h is an $(f-d)$ cap set for \overline{X}_h . By the topological uniqueness of $(f-d)$ cap sets [Ch₁, Theorem 6.2], $(\overline{X}_h, X_h) \cong (\tilde{X}, X)$. This completes the proof. \square

The l_2 -version of the above theorem is proved by a different method, in such a case we do not have to impose any condition on h :

3.8. Theorem. *Let X be an $(f-d)$ cap set for an l_2 -manifold \tilde{X} . Then each $h \in H(X)$ is conjugate to a $g \in H(X)$ which extends to $\tilde{g} \in H(\tilde{X})$.*

Proof. Using the Lavrentieff Theorem, there exist G_δ -sets G and G' in \tilde{X} containing X such that h extends to a homeomorphism $\tilde{h}: G \rightarrow G'$. Let $G_0 = G \cap G'$ and $G_n = \tilde{h}^{-1}(G_{n-1}) \cap \tilde{h}(G_{n-1})$ for $n = 1, 2, \dots$. Then $X' = \bigcap_{n=0}^{\infty} G_n$ is a G_δ -set in \tilde{X} containing X and $\tilde{h}(X') = X'$. Since $\tilde{X} \setminus X'$ is a countable union of Z -sets in \tilde{X} , X' is homeomorphic to \tilde{X} by [An]. Observe that X is an $(f-d)$ cap set for X' . By the topological uniqueness of $(f-d)$ cap sets [Ch₁], $(X', X) \cong (\tilde{X}, X)$, which implies the theorem. \square

Next, we will consider the case X is an l_2 -manifold and \tilde{X} is a Q -manifold such that $X^* = \tilde{X} \setminus X$ is a cap set for \tilde{X} .

3.9. Theorem. *Let X be an l_2 -manifold contained in a Q -manifold \tilde{X} with $X^* = \tilde{X} \setminus X$ a cap set for \tilde{X} and let $h \in H(X)$. Suppose there is a cap set Y for \tilde{X} with $Y \subset X^*$ and h extends to a proper map $\tilde{h}: \tilde{X} \rightarrow \tilde{X}$ such that $\tilde{h}(Y) = Y$ and $\tilde{h}|_Y \in H(Y)$. Then h is conjugate to a $g \in H(X)$ which extends to $\tilde{g} \in H(\tilde{X})$. (Cf. Question 5.1.)*

Proof. As seen before, $\overline{X}_h \cong \tilde{X}$ and $X_h \cong X$. Then X_h is a G_δ -set in \overline{X}_h , so $X_h^* = \overline{X}_h \setminus X_h$ is an F_σ -set in \overline{X}_h . Since X is map-dense in \tilde{X} , X_h is also map-dense in \overline{X}_h by Lemma 3.2. Hence X_h^* is a countable union of Z -sets in \overline{X}_h . We note that $\overline{Y}_{\tilde{h}|_Y} = \overline{X}_h$. As shown in the proof of Theorem 3.7, $Y_{\tilde{h}|_Y}$ is a cap set for \overline{X}_h . Since $Y_{\tilde{h}|_Y} \subset X_h^*$, X_h^* is also a cap set for \overline{X}_h by [Ch₁, Theorem 6.6]. Again by the topological uniqueness of cap sets, $(\overline{X}_h, X_h^*) \cong (\tilde{X}, X^*)$. Hence $(\overline{X}_h, X_h) \cong (\tilde{X}, X)$. This completes the proof. \square

Applying Theorems 3.3, 3.5 and 3.7, we have an answer for Question (A) in case X is a noncomplete manifold admitting a compact manifold completion.

3.10. Corollary. *Let $X = (X, d)$ be a metric space. Then each $h \in H(X) \cap C_U(X)$ is conjugate to a $g \in H_U(X)$ in the following cases:*

- (1) X is an n -manifold ($n \neq 4, 5$) with compact n -manifold completion;

(2) X is a Q -manifold with compact Q -manifold completion \tilde{X} such that $\tilde{X} \setminus X$ is a Z -submanifold;

(3) X is a Σ - (or σ -) manifold with compact Q -manifold completion \tilde{X} such that X is an $(f-d)$ cap set for \tilde{X} .

4. STABLE HOMEOMORPHISMS

In this section we will consider homeomorphisms of the form (or conjugate to) $h \times \text{id}$.

4.1. Lemma. *Let Y be a locally compact separable AR. Then the one-point compactification $\alpha(Y \times [0, 1)) = Y \times [0, 1) \cup \{\infty\}$ of $Y \times [0, 1)$ is an AR and $\{\infty\}$ is a Z -set in $\alpha(Y \times [0, 1))$.*

Proof. Let $\alpha(Y) = Y \cup \{*\}$ denote the one-point compactification of Y . We can regard $\alpha(Y \times [0, 1))$ as the quotient space

$$\alpha(Y) \times I / \{*\} \times I \cup \alpha(Y) \times \{1\}.$$

Let $q: \alpha(Y) \times I \rightarrow \alpha(Y \times [0, 1))$ be the identification map. The deformation of $\alpha(Y) \times I$ pushing up along the I -coordinates induces a contraction of $\alpha(Y \times [0, 1))$ onto $\{\infty\}$ fixing $\{\infty\}$. Then $\{\infty\}$ is a strong deformation retract of $\alpha(Y \times [0, 1))$. By the result of Kruse and Liebnitz [KL], $\alpha(Y \times [0, 1))$ is an AR. For any open cover \mathcal{U} of $\alpha(Y \times [0, 1))$, choose $U \in \mathcal{U}$ such that $\infty \in U$. Let V and W be open neighborhoods of $*$ in $\alpha(Y)$ such that $V \times I \subset q^{-1}(U)$ and $\text{cl } W \subset V$ and $k: \alpha(Y) \rightarrow [0, 1]$ a Urysohn map such that $k(\text{cl } W) = 1$ and $k(\alpha(Y) \setminus V) = 0$. Define a map $r: \alpha(Y) \times I \rightarrow \alpha(Y) \times I$ as follows:

$$r(y, t) = \begin{cases} (y, t) & \text{if } t \geq k(y), \\ (y, k(y)) & \text{if } t < k(y). \end{cases}$$

Next choose $0 < t_0 < t_1 < 1$ so that $\alpha(Y) \times [t_0, 1] \subset q^{-1}(U)$. Let $\varphi: Y \times I \rightarrow Y$ be a contraction with $\varphi_1(Y) = y_0$. Define a map $f: r(\alpha(Y) \times I) \rightarrow Y \times [0, 1)$ as follows:

$$f(y, t) = \begin{cases} (y_0, t_1) & \text{if } t_1 \leq t \leq 1, \\ \left(\varphi \left(y, \frac{t - t_0}{t_1 - t_0} \right), t \right) & \text{if } t_0 \leq t \leq t_1, \\ (y, t) & \text{if } 0 \leq t \leq t_0. \end{cases}$$

Clearly, $fr: \alpha(Y) \times I \rightarrow Y \times [0, 1)$ induces a map $g: \alpha(Y \times [0, 1)) \rightarrow Y \times [0, 1)$ fixing $(Y \setminus V) \times [0, t_0]$. Since

$$\alpha(Y \times [0, 1)) \setminus (Y \setminus V) \times [0, t_0] \subset U,$$

g is \mathcal{U} -close to id . Then $\{\infty\}$ is a Z -set in $\alpha(Y \times [0, 1))$. \square

4.2. Lemma. *Let X be an l_2 -manifold. If X is homeomorphic to the product $Y \times Z$ of a locally compact AR Y with Z , then Z is homeomorphic to X .*

Proof. Since $X \approx Y \times Z$ satisfies the discrete approximation property (DAP) $[\text{To}_1]$ and Y is locally compact, Z satisfies DAP. Since Z is a separable

complete metrizable ANR, it is an l_2 -manifold [To₁]. Thus the projection $p: Y \times Z \rightarrow Z$ is a fine homotopy equivalence (since Y is an AR) between l_2 -manifolds, hence a near homeomorphism [Fe, Theorem 3.4]. \square

4.3. Theorem. *Let X be an l_2 -manifold contained in a Q -manifold \tilde{X} such that $X^* = \tilde{X} \setminus X$ is a cap set for \tilde{X} and let $h \in H(X)$. Suppose there is a homeomorphism $\varphi: X \rightarrow Y \times Z$ of X onto the product of a locally compact AR Y with Z and an $f \in H(Y)$ such that $\varphi h \varphi^{-1} = f \times \text{id}$:*

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ \varphi \downarrow & & \downarrow \varphi \\ Y \times Z & \xrightarrow{f \times \text{id}} & Y \times Z \end{array}$$

Then h is conjugate to $g \in H(X)$ which extends to $\tilde{g} \in H(\tilde{X})$.

Proof. By Lemma 4.2, we may replace Z by X . Since $X \cong X \times [0, 1)$, we have a homeomorphism $\psi: Y \times X \rightarrow Y \times [0, 1) \times X$ with the commutative diagram:

$$\begin{array}{ccc} Y \times X & \xrightarrow{f \times \text{id}} & Y \times X \\ \psi \downarrow & & \downarrow \psi \\ Y \times [0, 1) \times X & \xrightarrow{f \times \text{id} \times \text{id}} & Y \times [0, 1) \times X \end{array}$$

Let $Y' = \alpha(Y \times [0, 1)) = Y \times [0, 1) \cup \{\infty\}$ be the one-point compactification of $Y \times [0, 1)$. Then $f \times \text{id} \in H(Y \times [0, 1))$ extends to $f' \in H(Y')$ with $f'(\infty) = \infty$. Clearly $f' \times \text{id} \in H(Y' \times X)$ extends to $f' \times \text{id} \in H(Y' \times \tilde{X})$. Since Y' is a compact AR by Lemma 4.1 and \tilde{X} is a Q -manifold, we have $Y' \times \tilde{X} \cong \tilde{X}$. It is easy to see that $Y' \times X^*$ is a cap set for $Y' \times \tilde{X}$. Since $\{\infty\} \times \tilde{X}$ is a Z -set in $Y' \times \tilde{X}$ by Lemma 4.1,

$$Y' \times \tilde{X} \setminus Y \times [0, 1) \times X = Y' \times X^* \cup \{\infty\} \times \tilde{X}$$

is a cap set for $Y' \times \tilde{X}$ [Ch₁, Theorem 6.6]. By the topological uniqueness of cap sets,

$$(Y' \times \tilde{X}, Y \times [0, 1) \times X) \cong (\tilde{X}, X).$$

This implies the theorem. \square

By the same proof, we have the following theorem.

4.4. Theorem. *Let X be an $(f$ -d) cap set for a Q -manifold \tilde{X} and $h \in H(X)$. Suppose there is a homeomorphism $\varphi: X \rightarrow Y \times X$ of X onto a product of a (finite-dimensional) locally compact AR Y with X and an $f \in H(Y)$ such*

that $\phi h \phi^{-1} = f \times \text{id}$:

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ \phi \downarrow & & \downarrow \phi \\ Y \times X & \xrightarrow{f \times \text{id}} & Y \times X \end{array}$$

Then h is conjugate to a $g \in H(X)$ which extends to $\tilde{g} \in H(\tilde{X})$.

Remark. By examples of Henderson and Walsh [HW], Lemma 4.2 is not true in case X is a Σ - (or σ -) manifold. We do not know whether X of $Y \times X$ in the above theorem can be replaced by another space Z .

4.5. Theorem. Let X be an l_2 -manifold contained in a Q -manifold \tilde{X} such that $\tilde{X} \setminus X$ is a cap set for \tilde{X} . If $h \in H(X)$ extends to a proper map $\tilde{h}: \tilde{X} \rightarrow \tilde{X}$, then $h \times \text{id} \in H(X \times s)$ is conjugate to a $g \in H(X \times s)$ which extends to $\tilde{g} \in H(\tilde{X} \times Q)$.

Proof. Let Y_0 be a cap set for X . Then $Y = \bigcup_{n \in \mathbb{Z}} h^n(Y_0)$ is also a cap set for \tilde{X} [Ch₁, Theorems 6.3 and 6.6]. By [Ch₁, Theorem 6.8], $Y \times (Q \setminus s)$ is a cap set for $\tilde{X} \times Q$. Since $h \times \text{id} \in H(X \times s)$ extends to a proper map $\tilde{h} \times \text{id}: \tilde{X} \times Q \rightarrow \tilde{X} \times Q$ such that $\tilde{h} \times \text{id}|_{Y \times (Q \setminus s)} \in H(Y \times (Q \setminus s))$, the result follows from Theorem 3.9. \square

As a consequence of Theorems 4.3 and 4.5, we have an answer for Question (5) in [AB] ([AK, CSQ 4]):

4.6. Corollary. Let $h: s \rightarrow s$ be a homeomorphism of the pseudo-interior of the Hilbert cube Q onto itself. In the following cases, h is conjugate to a $g \in H_U(s)$ (which extends to $\tilde{g} \in H(Q)$).

(1) There is a homeomorphism $\phi: s \rightarrow Y \times Z$ of s onto the product of a locally compact AR Y with another space Z and an $f \in H(Y)$ such that $\phi h \phi^{-1} = f \times \text{id}$:

$$\begin{array}{ccc} s & \xrightarrow{h} & s \\ \phi \downarrow & & \downarrow \phi \\ Y \times Z & \xrightarrow{f \times \text{id}} & Y \times Z \end{array}$$

(2) There is a homeomorphism $\phi: s \rightarrow s \times s$ and an $f \in H(s) \cap C_U(s)$ such that $\phi h \phi^{-1} = f \times \text{id}$:

$$\begin{array}{ccc} s & \xrightarrow{h} & s \\ \phi \downarrow & & \downarrow \phi \\ s \times s & \xrightarrow{f \times \text{id}} & s \times s \end{array}$$

Remark. The σ - or Σ -version of the above also holds. For (2), we have already a better result in Corollary 3.10.

Let $\dot{I}^n = I^n \setminus \partial I^n$ denote the interior of I^n . The following is a finite-dimensional version of the above corollary.

4.7. Corollary. *Let $h: \dot{I}^n \rightarrow \dot{I}^n$ ($n > 1$) be a homeomorphism which is conjugate to the form $f \times \text{id}$, where $f \in H(\dot{I}^{n-1})$. Then h is conjugate to a $g \in H_U(\dot{I}^n)$ (which extends to $\tilde{g} \in H(I^n)$).*

Proof. This follows from the fact

$$(I^n, \dot{I}^n) \cong (I^n / \partial I^{n-1} \times I \cup I^{n-1} \times \{1\}, \dot{I}^n). \quad \square$$

Remark. Each homeomorphism $h: \dot{I} \rightarrow \dot{I}$ is clearly a uniform homeomorphism.

5. OPEN QUESTIONS

5.1. If X is an l_2 -manifold with the compact Q -manifold completion \tilde{X} such that $\tilde{X} \setminus X$ is a cap set, is every $h \in H(X) \cap C_U(X)$ conjugate to a $g \in H_U(X)$?

5.2. Is every homeomorphism of s onto s conjugate to a uniform homeomorphism of s onto s ?

5.3. We do not know the answer if s in the above question is replaced by X , where $X = l_2$, \mathbf{R}^n , $n \geq 2$ or a Banach space.

5.4. Let $h: \dot{B}^2 \rightarrow \dot{B}^2$ be the homeomorphism constructed in Example 3.4. Given $X = \dot{B}^2 \times \dot{B}^2 \times \cdots$ the usual product metric, is the homeomorphism $h \times h \times \cdots$ conjugate to a uniform homeomorphism on X ? If the answer is "no", then it is a counterexample to 5.2.

Added in proof. Recently J. van Mill has constructed a counterexample to Question 5.2 (i.e. [AB, Question (5), AK, CSQ4]) in his paper: *A homeomorphism on s not conjugate to an extendable homeomorphism*.

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